Workshop on spectral measures: Functional Calculus & Spectral theorem for self-adjoint unbounded operators

Neba Polneau[1]

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[1]PhD student at IMJ-Sorbonne Univetsity, polneau@imj-prg.fr

Contents

0.1 Presentation of the workshop

This workshop was organized by Baptiste Dugue[1]. The goal is to understand the different aspects of spectral measures. Originally, Baptiste is interested in these tools for studying mixing properties of the geodesics flow on non-compact hyperbolic manifolds. In fact, some arguments using spectral theory was recently used to study such problems of dynamical systems questions. Therefore, we revisited some general aspects of spectral theory. The workshop was organized as a series of talks according to the interests of the audience. Here are the different talks given during the workshop.

1. *Introduction:*
   - Baptiste Dugue, *Mixing of the geodesics flows on hyperbolic surfaces.*
   - Neba Polneau, *Spectral measures & distorted Fourier transform.*
2. Paul Lebreton, *Continuous functional calculus.*
3. Marius Nietschmann, *Introduction to compact operators.*
4. Tristan Humbert, *Unbounded operators.*
5. Neba Polneau, *Functional Calculus & Spectral theorem for self-adjoint unbounded operators.*
6. Mario Gauvrit - TBA

In the sequel, we expose the notes of the talks 1 and 5 made by the author.

[1] https://perso.eleves.ens-rennes.fr/people/baptiste.dugue/

Chapter 1

Motivation through Schrödinger operators and soliton Dynamics

In this chapter, we motivate some interests for spectral measures as a tool to construct and understand Schrödinger operators. This understanding is made via what is called distorted Fourier transforms (dFT). The spectral measures associated to Schrödinger operators appeared as the object of the situation. Since our goal is to give general ideas, we do not provide rigorous proofs. The reader can see this chapter as a survey for introducing the topic.

1.1 Distorted Fourier transform & spectral measures

This section is devoted to the construction of a distorted Fourier transform.

1.1.1 Spectral theorem & spectral resolution

In the finite dimensional case, a self-adjoint matrix $M \in S_n(\mathbb{C})$ can be unitarily diagonalized as follows :

$$M = U^{-1}DU \tag{1.1}$$

with $U \in U_n(\mathbb{C})$ and D is a real diagonal matrix. In terms of operators, we have the following spectral resolution

$$M = \sum_{j=1}^{n} \lambda_j P_j \tag{1.2}$$

where λ_j are the eigenvalues of M (counted without multiplicity) and P_j the orthogonal projection onto the Eigenspace associated to λ_j.

For infnite-dimensional operators, the idea can be stated roughly as follows: For an unbounded self-adjoint operators $(A, D(A))$ on a Hilbert space $\mathscr{H}$, there exist a Borel set B, a measure μ on B and a real-valued function a on B, such that

$$A = U^{-1} M_a U \tag{1.3}$$

where M_a is the multiplication operator by a, $U : \mathscr{H} \to L^2(B, d\mu)$ is a unitary isomorphism. This diagonalization gives a spectral decomposition of A in analogy to the case of matrices (1.1). We can also do an analogy of the spectral resolution (1.2). Roughly speaking, the operator A can be decomposed through each "spectral pieces" via an operator-valued integral as follows

$$A = \int_{\sigma(A)} \lambda \, dP_\lambda \ . \tag{1.4}$$

We can understand dP_λ as a projection-valued measure defined by

$$\int_{\lambda_1}^{\lambda_2} dP_\lambda = P_{\lambda_1} - P_{\lambda_2}$$

with P_λ a family of orthogonal projector. At this stage, spectral measures can be introduced. In fact, for any $u \in \mathscr{H}$, $\langle dP_\lambda u, u\rangle$ is the spectral measure rigorously defined in (2.6).

1.1.2 Distorted Fourier transform

In this part, we work with Schrödinger operators in one dimension. We consider $V \in \mathscr{C}_c(\mathbb{R}, \mathbb{R})$ and we define the Schrödinger operator

$$H := -\partial_{xx} + V \qquad ; \qquad D(H) := H^2(\mathbb{R}) \tag{1.5}$$

We state the following lemma whose elements of proof can be found in standard spectral theory book (see [1], [3]).

Lemma 1.1.1. *The following statements hold:*

(i) The unbounded operator $(H, D(H))$ is self-adjoint.

(ii) The spectrum of H consists of :

- *an (absolutely) continuous part which is $[0, +\infty)$*

- *a finite number of negative eigenvalues with finite multiplicity.*

Remark 1.1.2. *The domain we use is not the only choice to construct self-adjoint Schrödinger operators. We can weaken the assumptions on V and still have a self-adjoint operator (see [3, Chapter 3]) . For instance:*

- *For $V \in L^2(\mathbb{R})$, we can take $D(H) = H^2(\mathbb{R})$. The self-adjointness follows from the perturbative method of Rellich-Kato.*
- *For $V \in L^1(\mathbb{R})$, we can take $D(H) = \{f \in H^1(\mathbb{R}) : -\partial_{xx} f + V f \in L^2(\mathbb{R})\}$. The self-adjointness is realized through Friedrich extension via quadratic form.*

Nevertheless, in such situations, infinitely many discrete negative eigenvalues can appear in the specrtrum.

Recalling the diagonalization statement (2.1), we can diagonalize $(H, D(H))$ as

$$H = \mathscr{F}^{-1} M_h \mathscr{F}$$

where $\mathscr{F}$ is a unitary isomorphism. Such unitary isomorphism are called *distorted Fourier transform* associated to H.

Example 1.1.3. ***The Laplacian on*** $\mathbb{R}$. *It correspond to the case where $V = 0$. We know from Fourier theory that the Fourier transform diagonalizes the Laplacian as follows*

$$-\partial_{xx} = \widehat{\mathscr{F}}^{-1} M_{k^2} \widehat{\mathscr{F}} \tag{1.6}$$

where $\widehat{\mathscr{F}}$ is the classical or flat Fourier transform

$$\widehat{\mathscr{F}} f(k) := \widehat{f}(k) := \int_{\mathbb{R}} e^{-ikx} f(x)\ dx\ . \tag{1.7}$$

We recall the fact that $\sigma(-\partial_{xx}) = [0, +\infty)$.

1. *Spectral resolution. For any $f \in \mathcal{S}(\mathbb{R})$, we can write from (1.6)*

$$\begin{aligned}
(-\partial_{xx} f)(x) &= \int_{\mathbb{R}} e^{ixk} k^2 \widehat{f}(k)\ \frac{dk}{2\pi} \\
&= \int_0^{+\infty} e^{ix\sqrt{\lambda}} \lambda \widehat{f}(\sqrt{\lambda})\ \frac{d\lambda}{2\pi\sqrt{\lambda}} + \int_0^{+\infty} e^{-ix\sqrt{\lambda}} \lambda \widehat{f}(-\sqrt{\lambda})\ \frac{d\lambda}{2\pi\sqrt{\lambda}} \\
&= \int_0^{+\infty} \lambda (dP_\lambda f)(x)
\end{aligned}$$

where

$$(dP_\lambda f)(x) = \left[e^{ix\sqrt{\lambda}} \widehat{f}(\sqrt{\lambda}) + e^{-ix\sqrt{\lambda}} \widehat{f}(-\sqrt{\lambda})\right] \frac{d\lambda}{2\pi\sqrt{\lambda}}\ .$$

2. *Spectral projectors. For $b > 0$, we can write*

$$
\begin{aligned}
(P_b f)(x) &= \int_0^b (dP_\lambda f)(x) \\
&= \int_0^b \left[e^{ix\sqrt{\lambda}} \widehat{f}(\sqrt{\lambda}) + e^{-ix\sqrt{\lambda}} \widehat{f}(-\sqrt{\lambda}) \right] \frac{d\lambda}{2\pi\sqrt{\lambda}} \\
&= \int_{-\sqrt{b}}^{\sqrt{b}} e^{ixk} \widehat{f}(k) \, \frac{dk}{2\pi} \, .
\end{aligned}
$$

Thus, we have

$$
P_b f = \widehat{\mathscr{F}}^{-1} M_{k \mapsto \mathbf{1}_{[0,b]}(k^2)} \widehat{\mathscr{F}} f =: \mathbf{1}_{[0,b]}(-\partial_{xx}) \, .
$$

Exercise 1.1.4. 1. ***Laplacian on*** $\mathbb{R}^d$. *Show that the spectral resolution of $-\triangle := \sum_{j=1}^d \partial_{x_j x_j}$ is given by : for $f \in \mathcal{S}(\mathbb{R}^d)$*

$$
(dP_\lambda f)(x) = \int_{\mathbb{S}^{d-1}} e^{i\sqrt{\lambda} k_0 \cdot x} \widehat{f}(\sqrt{\lambda} k_0) \frac{d\eta(k_0)}{(2\pi)^d} \, \lambda^{\frac{d-2}{2}} \, d\lambda \, .
$$

2. ***Laplacian on*** $\mathbb{H}^d$. *We consider the Poincaré Half space defined by*

$$
\mathbb{H}^d := \{(x_1, \cdots, x_d) : x_d > 0\}
$$

endowed withe the hyperbolic metric

$$
g_{\mathbb{H}^d} := \frac{dx_1^2 + \cdots + dx_d^2}{x_d^2}.
$$

What it is the spectral resolution of $\triangle_{\mathbb{H}^d}$?

From what happens for the Laplacian on $\mathbb{R}$, we can infer that the projector-valued spectral measure dP_λ is the "derivative" of the spectral projectors P_λ. In the case of the Laplacian, we already have the classical Fourier transform that helps build everything. Thus, what happens for about general Schrödinger operators. Here is a road map to understand and construct distorted Fourier transform:

1. Functional calculus for self-adjoint operators
2. spectral measures
3. spectral projectors and their relationship to spectral measures

4. How to compute spectral projectors in terms of resolvents (Stone's formula)

5. Linear scattering theory for Schrödinger operators.

Remark 1.1.5. *These ideas can be generalized to more general second order differential operators such as Sturm-Liouville operators (Weyl-Kodaira-Tichtmarsch theory).*

1.2 Link to soliton dynamics

Roughly speaking, solitons are ground states for semilinear dispersive PDE. One should keep in mind that the question of their stability is closely related to small data theory of nonlinear dispersive PDE involving a linear operators perturbed by a potential that depends on the soliton and the type of nonlinearity. In the case of dispersice PDEs such as nonlinear Schrödinger, Waves, Klein-Gordon, it is a Schrödinger opearator. Its spectrum gives at least heuristically the different issues for either stability or instability. Spectral measures give more precise information. In fact, It brings a refined decomposition of spectrum : absolutely continuous spectrum, pure point spectrum, ... we have the following issues:

1. The absolutely continuous spectrum is responsible for radiation phenomena leading to scattering and dispersive decay in time.

2. The pure point spectrum (which means eigenvalues) can exhibit different types of behavior depending on the model:

   - *Discrete spectrum.* The negative spectrum are, in general, reason for instability. Positive ones are called in some model, such as Klein-Gordon, internal modes. It leads to weaker radiation phenomena due to nonlinear interactions with the absolute continuous spectrum part.
   - *Embedded eigenvalues.* Those are eigenvalues not isolated to the continuous part. They are in general difficult to treat. They are linked to the Fermi golden rule.

Chapter 2

Functional Calculus for self-adjoint unbounded operators

We fix a separable Hilbert space $\mathscr{H}$. We consider a self-adjoint unbounded operator $(A, D(A))$ on $\mathscr{H}$. We denote $\mathscr{B}(\mathscr{H})$ the algebra of bounded operators on $\mathscr{H}$. We note $\mathscr{L}^\infty$ the space of bounded Borel functions.

2.1 Introduction

The goal of this chapter is essentially to prove the following theorem.

Theorem 2.1.1 ($\mathscr{L}^\infty$ functional calculus)**.** *There exist a unique map*

$$\begin{aligned} \Phi : \mathscr{L}^\infty(\mathbb{R}, \mathbb{C}) &\to \mathscr{B}(\mathscr{H}) \\ f &\mapsto f(A) \end{aligned}$$

such that :

(i) Φ *is a* $*-$*algebra morphism. i.e. for any* $\alpha \in \mathbb{C}$ *and* $f, g \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{C})$

- $(f + \alpha g)(A) = f(A) + \alpha g(A)$,
- $(fg)(A) = f(A)g(A)$,
- $\mathbf{1}(A) = Id_{\mathscr{H}}$,
- $\overline{f}(A) = f(A)^*$.

(ii) Φ *is continuous with* $\|f(A)\| \leq \|f\|_\infty$*, for all* $f \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{C})$.

(iii) For $z \in \mathbb{C}\backslash\mathbb{R}$*, we consider* $f_z : x \mapsto (x - z)^{-1}$*. Its image is given by* $f_z(A) = (A - z)^{-1}$.

(iv) *For $f \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{C})$, assume there exists a uniformly bounded sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathscr{L}^\infty(\mathbb{R}, \mathbb{C})$ such that f_n converges pointwise to f. Then, for any $u \in \mathscr{H}$, $f_n(A)u \to f(A)u$ when $n \to +\infty$.*

After proving this, we will extend it in a precise way to $\mathscr{L}^\infty_{\mathrm{loc}}$ functions. Before starting to prove the theorem, let us discuss some principles in constructing functional calculus. First, one way to construct functional calculus is to make it on certain large enough class of explicit functions and extend it. For bounded operators, this mechanism is quite clear. In fact, we know the functional calculus for functions $x \mapsto x^k$. Thus, we know for any polynomial by linearity of the functional calculus. Since the spectrum of bounded operators is compact, we can use Stone-Weierstrass to approximate any continuous function on the spectrum by polynomials. Then, we obtain functional calculus for continuous functions. This strategy can not be applied straightforwardly for the case of unbounded operators. Indeed, A^k has to be defined with care due to some domain issues when dealing with unbounded operators. In any case, the density argument is subtle since the spectrum of unbounded operators is not necessarily bounded. Nevertheless, an analogous strategy will be implemented using the resolvents. We also perform an approximation process. We do not use polynomials, but rather rational functions whose poles are not on the real line. We end up approximating $\mathscr{C}_0(\mathbb{R}, \mathbb{C})$ functions and we obtain functional calculus at least for this class of functions.

Second, having in mind spectral theorems, one can easily deduce functional calculus too. In fact, let us assume that we can represent A as multiplication operator as follows

$$A = U^{-1} M_a U \ . \tag{2.1}$$

M_a is the multiplication operator by a real-valued function a, the operator $U : \mathscr{H} \to L^2(B, d\mu)$ is a unitary isomorphism, where B a Borel set and μ a measure on B. We construct functional calculus by setting

$$f(A) = U^{-1} M_{f(a)} U \qquad \text{with } D(f(A)) = U^{-1} D(M_{f(a)}) \ .$$

But establishing such spectral result is as challenging as constructing functional calculus. In fact, both statements are equivalent as we will see. We establish spectral theorem by first constructing $\mathscr{C}_0$ functional calculus. All theses relationships rely on the notion of spectral measures.

2.2 $\mathscr{C}_{\mathrm{lim}}$ functional calculus

In this section, we construct the functional calculus for the class of functions $\mathscr{C}_{\mathrm{lim}} := \mathscr{C}_0 \oplus \mathbb{C}$. It is the space of continuous functions having the same limit at both $\pm\infty$. We consider the $*$-algebra $\mathscr{A}$ generated by $x \mapsto 1$ and the functions $x \mapsto (x-z)^{-1}$ for any $z \in \mathbb{C}\backslash\mathbb{R}$.

Lemma 2.2.1. *The following holds:*

(i) The $*-$algebra $\mathscr{A}$ is dense in $\mathscr{C}_{lim}(\mathbb{R},\mathbb{C})$ for the uniform norm.

(ii) The $*-$algebra $\mathscr{A}$ is stable by square root. i.e. for any $f \in \mathscr{A}$ such that $f \geq 0$, there exists $g \in \mathscr{A}$ such that $f = |g|^2$.

Proof. For (i), we use the locally-compact version of Stone-Weierstrass theorem (see [2], Theorem 5.4.3). In fact, this lead to the fact the $*$-algebra generated by the functions $x \mapsto (x-z)^{-1}$ for any $z \in \mathbb{C}\backslash\mathbb{R}$ is dense in $\mathscr{C}_0(\mathbb{R},\mathbb{C})$ for the uniform norm. We get the density of $\mathscr{A}$ in $\mathscr{C}_{\lim}(\mathbb{R},\mathbb{C})$ by just adding the constant functions.

For (ii), we consider $f \in \mathscr{A}$ such that $f \geq 0$. Then, f is a rational fraction that can be written in irreducible form as

$$f(x) = c\frac{P(x)}{Q(x)}$$

where $c \in \mathbb{C}$, the polynomials P, Q are prime with each other. Moreover, we have

$$\begin{aligned} P(x) &= \prod_{i=1}^{m}(x-\alpha_i)^{m_i}\prod_{j=1}^{p}(x-z_j)^{p_j} \ , \\ Q(x) &= \prod_{k=1}^{q}(x-z'_k)^{q_k} \ , \end{aligned} \tag{2.2}$$

where $\alpha_i \in \mathbb{R}$, $z_j, z'_k \in \mathbb{C}\backslash\mathbb{R}$.

Since f is real, i.e. $\overline{f(x)} = f(x)$ for any $x \in \mathbb{R}$, this gives

$$\overline{c}\,\overline{P(x)}Q(x) = cP(x)\overline{Q(x)} \ .$$

Thus, the expressions (2.2) yields

$$\overline{c}\prod_{j=1}^{p}(x-\overline{z_j})^{p_j}\prod_{k=1}^{q}(x-z'_k)^{q_k} = c\prod_{j=1}^{p}(x-z_j)^{p_j}\prod_{k=1}^{q}(x-\overline{z'_k})^{q_k} \ . \tag{2.3}$$

Then, by identifying the dominant coefficient of the polynomials in both sides of (2.3), we get c is real. Moreover, the fact that P and Q are prime with each other makes any z_j different to any z'_k. Thus, $\prod_{j=1}^{p}(x-\overline{z_j})^{p_j}$ divides $\prod_{j=1}^{p}(x-z_j)^{p_j}$ and $\prod_{j=1}^{p}(x-z_j)^{p_j}$ divides $\prod_{j=1}^{p}(x-\overline{z_j})^{p_j}$. This leads to the fact that the non-real root of P appeared with their conjugate with the same multiplicity. The same reasoning leads to the same result for Q. Thus,

$$P(x) = \prod_{i=1}^{m}(x-\alpha_i)^{m_i}\prod_{j=1}^{p/2}|x-z_j|^{2p_j} \ ,$$
$$Q(x) = \prod_{k=1}^{q/2}|x-z'_k|^{2q_k} \ .$$

Therefore, f is non-negative if and only if $x \mapsto c\prod_{i=1}^{m}(x-\alpha_i)^{m_i}$ is non-negative. This implies c is positive and $m_i = 2m'_i$ with $m'_i \in \mathbb{N}$. We set

$$g(x) := \sqrt{c}\frac{\prod_{i=1}^{m}(x-\alpha_i)^{m'_i}\prod_{j=1}^{p/2}(x-z_j)^{p_j}}{\prod_{k=1}^{q/2}(x-z'_k)^{q_k}}.$$

We have $g \in \mathscr{A}$ and $f = |g|^2$.

$\square$

Proposition 2.2.2. *There exist a unique map*

$$\begin{aligned}\Phi : \mathscr{C}_{lim}(\mathbb{R},\mathbb{C}) &\to \mathscr{B}(\mathscr{H})\\ f &\mapsto f(A)\end{aligned}$$

satisfying (i)*,* (ii) *and* (iii) *of Theorem 2.1.1.*

Proof. First, we construct the functional calculus on $\mathscr{A}$. Any $f \in \mathscr{A}$, as a rational fraction, admits a unique decomposition in simple elements. There exist $N \in \mathbb{N}$, $n_j \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $z_j \in \mathbb{C}\backslash\mathbb{R}$ such that

$$f(x) = c_0 + \sum_{j=1}^{N}\frac{c_j}{(x-z_j)^{n_j}} \ .$$

We set

$$f(A) := c_0 Id_{\mathscr{H}} + \sum_{j=1}^{N}c_j(A-z_j)^{-n_j} \ . \tag{2.4}$$

The properties of (i) follows from the fact the rational fraction calculus is the same as the calculus in the resolvents set $R_A := \{f(A) : f \in \mathscr{A}\}$. We let the verification as an exercise for the reader (see Exercise 2.2.3). The property (iii) is straightforward.

We focus on the proof of (ii). For any $f \in \mathscr{A}$, we know that $\|f\|_\infty^2 - |f|^2 \in \mathscr{A}$ and $\|f\|_\infty^2 - |f|^2 \geq 0$. Then, Lemma 2.2.1 implies the existence of $g \in \mathscr{A}$ such that $\|f\|_\infty^2 - |f|^2 = |g|^2$. Thus, for any $u \in \mathscr{H}$, we have

$$\left\langle (\|f\|_\infty^2 - |f|^2)(A)u, u \right\rangle = \left\langle |g|^2(A)u, u \right\rangle = \left\langle g^*(A)g(A)u, u \right\rangle = \left\langle g(A)u, g(A)u \right\rangle .$$

We get

$$\left\langle (\|f\|_\infty^2 - |f|^2)(A)u, u \right\rangle \geq 0 . \tag{2.5}$$

On the other and, we have

$$\left\langle (\|f\|_\infty^2 - |f|^2)(A)u, u \right\rangle = \|f\|_\infty^2 \|u\|^2 - \|f(A)u\|^2 .$$

Combining this with (2.5), we get for any $u \in \mathscr{H}$,

$$\|f\|_\infty^2 \|u\|^2 \geq \|f(A)u\|^2 .$$

We get the $\mathscr{C}_{\lim}$ functional calculus by density due to Lemma 2.2.1.

□

Exercise 2.2.3. *Justify the fact the functional calculus on $\mathscr{A}$ constructed at (2.4) satisfy the properties of (i) of Theorem 2.1.1.*

2.3 $\mathscr{L}^\infty$ functional calculus

2.3.1 Spectral measures

After constructing the $\mathscr{C}_{\lim}$ functional calculus, we would like to extend it to bounded Borel functions. This extension process is performed with respect to how we approximate functions. In the previous section, we use the uniform norm. Since the uniform limit of continuous functions is still continuous, it is clear that we can not get bigger space than $\mathscr{C}_{\lim}(\mathbb{R}, \mathbb{C})$. Therefore, we need to weaken the topology we use for the approximation. For that sake, we use the topology of dominated convergence. It means convergence in a certain measure space which involved the so called spectral measures.

Proposition 2.3.1. *Let $u \in \mathscr{H}$.*

(i) There exist a unique finite positive regular Borel measure μ_u^A such that for all $f \in \mathscr{C}_0(\mathbb{R}, \mathbb{C})$

$$\langle f(A)u, u \rangle = \int_{\mathbb{R}} f(x)\ d\mu_u^A(x) . \tag{2.6}$$

The measure μ_u^A is called the spectral measure associated to A and u.

(ii) Moreover,

$$\mu_u^A(\mathbb{R}) = \|u\|^2 \ . \tag{2.7}$$

The relation (2.6) extends to the $\mathscr{C}_{lim}$ function.

Proof. For (i), we consider the following linear form

$$\begin{aligned} \omega_u : \mathscr{C}_0(\mathbb{R}, \mathbb{C}) &\to \mathbb{C} \\ f &\mapsto \langle f(A)u, u\rangle \ . \end{aligned}$$

Proposition 2.2.2 ensures the continuity of this linear form. Moreover, if $f \in \mathscr{C}_0$ is non-negative, we have that $\sqrt{f} \in \mathscr{C}_0$. Thus,

$$\langle f(A)u, u\rangle = \langle \sqrt{f}(A)u, \sqrt{f}(A)u\rangle \geq 0 \ .$$

Hence, the existence of the claimed spectral measure follows from the Riesz-Markov representation theorem (see [4, Theorem 2.14]) . It remains to show that this measure is finite. It suffices to show that for any $a, b \in \mathbb{R}$, $\mu_u^A([a,b]) \leq \|u\|^2$. Then, the result follows from the interior regularity of the spectral measure. We consider the piecewise continuous approximation of $\mathbf{1}_{[a,b]}$ as illustrated in the figure beneath. We can note that φ_n approximate $\mathbf{1}_{[a,b]}$ pointwisely and $\varphi_n \leq \varphi_1$. In addition, we have $\varphi_1 \in L^1(\mathbb{R}, d\mu_u^A)$. Indeed, since $\varphi_1 \in \mathscr{C}_0$, we can write

$$\int_{\mathbb{R}} |\varphi_1(x)| \ d\mu_u^A(x) = \langle |\varphi_1|(A)u, u\rangle \leq \|\varphi_1\|_\infty \|u\|^2 = \ \|u\|^2.$$

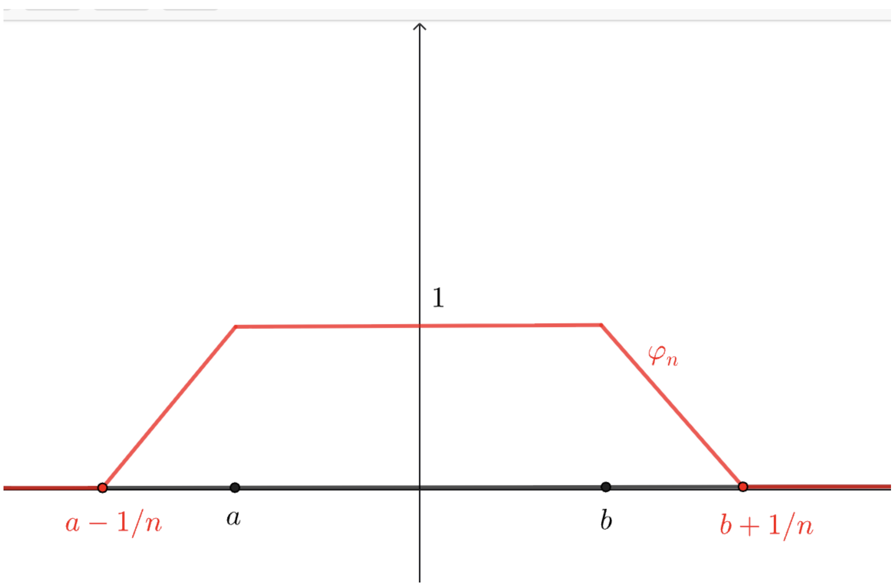

Figure 2.1: Approximation of $\mathbf{1}_{[a,b]}$

Thus, we can apply dominated convergence to obtain

$$\begin{aligned}
\mu_u^A([a,b]) = \int_{\mathbb{R}} \mathbf{1}_{[a,b]}(x)\ d\mu_u^A(x) &= \lim_{n\to+\infty} \int_{\mathbb{R}} \varphi_n(x)\ d\mu_u^A(x) \\
&= \lim_{n\to+\infty} \langle \varphi_n(A)u, u\rangle \\
&\leq \lim_{n\to+\infty} \|\varphi_n\|_\infty \|u\|^2 = \|u\|^2\ .
\end{aligned}$$

For (ii), we do a pointwise approximation of the indicator function $\mathbf{1}_{\mathbb{R}}$ with the sequence $\psi_n : x \to \frac{in}{in+x} \in \mathscr{A} \cap \mathscr{C}_0$. Since the sequence of functions ψ_n is uniformly bounded, the finiteness of the spectral measure allows to use dominated convergence. Thus, we can write

$$\begin{aligned}
\int_{\mathbb{R}} d\mu_u^A(x) &= \lim_{n\to+\infty} \int \psi_n(x)\ d\mu_u^A(x) \\
&= \lim_{n\to+\infty} \langle \psi_n(A)u, u\rangle \\
&= \lim_{n\to+\infty} \langle in(A+in)^{-1}u, u\rangle \\
&= \langle u, u\rangle\ .
\end{aligned}$$

This gives (2.7). Thus, for any function $f \in \mathscr{C}_0$ and any constant $c \in \mathbb{C}$, we have

$$\begin{aligned}
\langle (f+c)(A)u, u\rangle = \langle f(A)u, u\rangle + c\langle u, u\rangle &= \int_{\mathbb{R}} f(x)\ d\mu_u^A(x) + c\int_{\mathbb{R}} d\mu_u^A(x) \\
&= \int_{\mathbb{R}} (f(x)+c)\ d\mu_u^A(x)\ .
\end{aligned}$$

□

Exercise 2.3.2. *Justify the fact that $in(A+in)^{-1}u$ converges to u when $n \to +\infty$.*

2.3.2 Construction of the $\mathscr{L}^\infty$ functional calculus

Let $f \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{C})$. We know that f is a pointwise limit of a uniformly bounded sequence of $\mathscr{C}_{\lim}$ functions $(f_n)_{n\in\mathbb{N}}$. Thus, we define for any $u \in \mathscr{H}$,

$$f(A)u := \lim_{n\to+\infty} f_n(A)u \tag{2.8}$$

Proposition 2.3.3. *Let $f \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{C})$. The following holds :*

(i) For any $u \in \mathscr{H}$, the element $f(A)u$ as constructed above is well defined. i.e The limit exists and does not depend on the choice of the sequence f_n.

(ii) Moreover, for any $u \in \mathscr{H}$,

$$\langle f(A)u, u\rangle = \int_{\mathbb{R}} f(x)\ d\mu_u^A(x)\ . \tag{2.9}$$

(iii) $f \mapsto f(A)$ is the unique $\mathscr{L}^\infty$ functional calculus claimed in Theorem 2.1.1.

Proof. For (i), we consider a sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{C}_{\lim}^{\mathbb{N}}$ such that there exist $C > 0$ such that for all $n \in \mathbb{N}$, $|f_n| \le C$ and for all $x \in \mathbb{R}$, $f_n(x)$ converges to $f(x)$ when $n \to +\infty$. We show that $(f_n)(A)u$ is a Cauchy sequence in $\mathscr{H}$. Indeed, we have for $n, m \in \mathbb{N}$,

$$\|f_n(A)u - f_m(A)u\|^2 = \langle |f_n - f_m|^2(A)u, u\rangle = \int_{\mathbb{R}} |f_n(x) - f_m(x)|^2\ d\mu_u^A(x)\ .$$

Since μ_u^A is a finite measure, dominated convergence theorem yields f_n converges to f in $L^2(\mathbb{R}, d\mu_u^A)$. This combined with the previous equality implies that $f_n(A)u$ is a Cauchy sequence. Now, to prove that the limit does not depend on the sequence, we choose two such sequences f_n and g_n. Thus, the difference $f_n - g_n$ is uniformly bounded and converges pointwise to 0. Then, dominated convergence implies that $f_n - g_n$ converges to 0 in $L^2(\mathbb{R}, d\mu_u^A)$. Since

$$\|f_n(A)u - g_n(A)u\|^2 = \int_{\mathbb{R}} |f_n(x) - g_n(x)|^2\ d\mu_u^A(x)\ ,$$

we get the desired result.

For (ii), we consider the approximating $\mathscr{C}_{\lim}$ functions f_n as above. We know that

$$\langle f_n(A)u, u\rangle = \int_{\mathbb{R}} f_n(x)\ d\mu_u^A(x)\ .$$

Passing to the limit when $n \to \infty$, the right hand side gives $\langle f(A)u, u\rangle$. The left hand side gives $\int_{\mathbb{R}} f(x)\ d\mu_u^A(x)$ by dominated convergence theorem.

For (iii), The statements (ii) and (iii) of Theorem 2.1.1 are obvious, because the $\mathscr{L}^\infty$, we constructed coincide with the $\mathscr{C}_{\lim}$ functional calculus constructed in previous sections. For the statement (i) of Theorem 2.1.1, let us consider $f, g \in \mathscr{L}^\infty$. We take uniformly bounded sequences of $\mathscr{C}_{\lim}$ functions f_n and g_n approximating respectively f and g pointwise. We have $f_n + g_n \in \mathscr{C}_{\lim}$ uniformly bounded that approximates $f + g$ pointwise. Thus

$$(f+g)(A)u = \lim_{n\to\infty} (f_n + g_n)(A)u = \lim_{n\to\infty} f_n(A)u + g_n(A)u = f(A)u + g(A)u\ .$$

We also have $f_n g_n \in \mathscr{C}_{\lim}$ uniformly bounded that approximates fg pointwise. Thus

$$(fg)(A)u = \lim_{n\to\infty} (f_n g_n)(A)u = \lim_{n\to\infty} f_n(A) g_n(A)u = f(A)g(A)u\ .$$

Similar reasoning leads to $\overline{f}(A) = f(A)^*$.

It remains to prove the statement (iv) of Theorem 2.1.1. We consider a sequence $(f_n)_{n\in\mathbb{N}} \in \mathscr{C}_{\lim}^{\mathbb{N}}$ such that there exist $C > 0$ such that for all $n \in \mathbb{N}$, $|f_n| \leq C$ and for all $x \in \mathbb{R}$, $f_n(x)$ converges to $f(x)$ when $n \to +\infty$. We can write

$$\|f_n(A)u - f(A)u\|^2 = \langle |f_n - f|^2(A)u, u\rangle .$$

Invoking (2.9), we have

$$\|f_n(A)u - f(A)u\|^2 = \int_{\mathbb{R}} |f_n(x) - f(x)|^2 \, d\mu_u^A(x)$$

that converges to 0 by dominated convergence. Then, the uniqueness of the $\mathscr{L}^\infty$ functional calculus follows from this last property. In fact, it tells us that the one we constructed at (2.8) is the only one satisfying all the statements of Theorem 2.1.1. □

2.3.3 Comments on constructing functional calculus

We constructed $\mathscr{L}^\infty$ functional calculus starting from the Resolvents algebra to then reach the $\mathscr{C}_{\lim}$ functional calculus. This process succeeded because we have a natural formula for $f_z(A)$ when $f_z(x) = (x - z)^{-1}$ with $z \in \mathbb{C}\backslash\mathbb{R}$. It was given by the resolvents $(A - z)^{-1}$. In other framework, we can perform a similar idea using the continuous group of unitary operators $e^{itA} := U(t)$ given by the Stone's theorem (see [1, Theorem 7.12]. It is a 1-parameter group of operators satisfying $U(t + s) = U(t)U(s)$ for any $t, s \in \mathbb{R}$. It also satisfies the evolution equation

$$\begin{cases} \frac{d}{dt}(U(t)v) = iAU(t)v = iU(t)Av \ , \\ U(0) = v \in D(A) \ . \end{cases}$$

The operator iA is the generator of this group of operators. i.e. for all $v \in D(A)$,

$$iAv = \lim_{t\to 0} \frac{e^{itA}v - v}{t} \quad \text{in } \mathscr{H} \ .$$

Then we construct a functional calculus by mimicking the Fourier inversion formula on the Schwartz class $\mathcal{S}(\mathbb{R})$. In fact, we consider the Fourier transform

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx} \, dx \ , \qquad \text{for } f \in \mathcal{S}(\mathbb{R}) \ .$$

We have the inversion formula

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t)e^{itx} \, \frac{dt}{2\pi} \ . \tag{2.10}$$

Then, for any $f \in \mathcal{S}(\mathbb{R})$, we define

$$f(A) = \int_{\mathbb{R}} \widehat{f}(t)e^{itA}\ \frac{dt}{2\pi}\ . \tag{2.11}$$

Using the property of the Fourier transform we can construct $\mathcal{S}(\mathbb{R}) \oplus \mathbb{C}$ funtional calculus and extend it to $\mathscr{C}_{\lim}$ functional calculus. The interested reader can consult [1, Chapter 8] for all the details.

2.4 $\mathscr{L}^{\infty}_{\text{loc}}$ functional calculus

The $\mathscr{L}^{\infty}$ functional calculus produced bounded operators, therefore we didn't have to deal with domain issues. But, this question of domain has to be take into account when dealing with $\mathscr{L}^{\infty}_{\text{loc}}$ functions. For instance, if we construct such functional calculus, it reasonable to expect $(x \mapsto x)(A) = A$ which is unbounded.

Let $f \in \mathscr{L}^{\infty}_{\text{loc}}(\mathbb{R}, \mathbb{C})$. To resolve this issue, we consider the following space

$$D_f = \left\{ u \in \mathscr{H} : \int_{\mathbb{R}} |f(x)|^2\ d\mu_u^A(x) < +\infty \right\}\ . \tag{2.12}$$

For any $u \in D_f$, since $f \in L^2(\mathbb{R}, d\mu_u^A)$, there exists a sequence of $\mathscr{L}^{\infty}$ functions such that $f_n \to f$ in $L^2(\mathbb{R}, d\mu_u^A)$. For example, we can take $f_n := f\mathbf{1}_{f \leq n}$. Thus, we define

$$f(A)u := \lim_{n \to +\infty} f_n(A)u \tag{2.13}$$

Proposition 2.4.1. *Let $f \in \mathscr{L}^{\infty}_{loc}(\mathbb{R}, \mathbb{C})$. The following holds :*

(i) For any $u \in D_f$, the element $f(A)u$ as constructed above is well defined. i.e the limit exists and does not depend on the choice of the sequence f_n.

(ii) The unbounded operator $f(A)$ with domain $D(f(A)) = D_f$ gives a $\mathscr{L}^{\infty}_{loc}$ functional calculus. i.e. It coincides with the $\mathscr{L}^{\infty}$ functional calculus, all the algebraic properties (i) of Theorem 2.1.1 are satisfied as long as there is no domain issues. For $g \in \mathscr{L}^{\infty}_{loc}(\mathbb{R}, \mathbb{C})$,

- *If $u \in D(f(A)) \cap D(g(A))$, then $u \in D((f+g)(A))$ and*

$$(f+g)(A)u = f(A)u + g(A)u\ . \tag{2.14}$$

- *For all $u \in \{D(g(A)) : g(A)u \in D(f(A))\}$, it holds that*

$$(fg)(A)u = f(A)g(A)u\ . \tag{2.15}$$

- *It holds that $D(\overline{f}(A)) = D(f(A))$ and $\overline{f}(A) = f(A)^*$.*
- *Moreover, for any $u \in D(f(A))$,*

$$\langle f(A)u, u\rangle = \int_{\mathbb{R}} f(x)\ d\mu_u^A(x)\ . \tag{2.16}$$

Proof. We let it as an exercise for the reader since the arguments are similar to the proof of Proposition 2.3.3 (See Exercise 2.4.2). □

Exercise 2.4.2. *Prove Proposition 2.4.1.*

We end this chapter with the following diagram. It summarizes the discussion about constructing functional calculus. The part in red will be the topic of the next chapter.

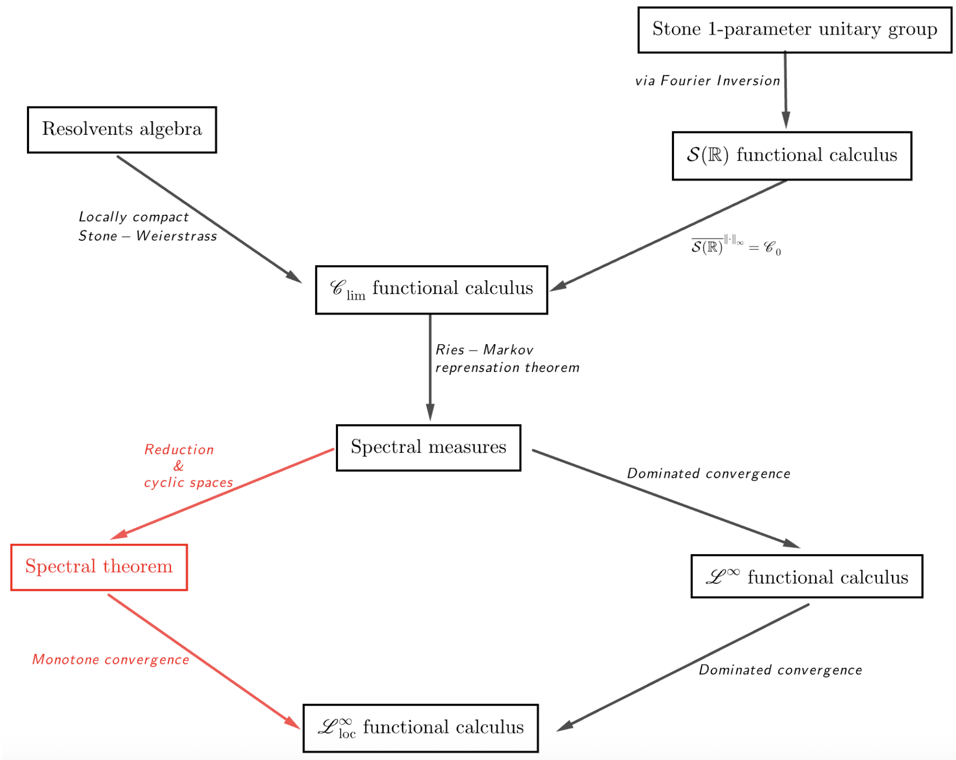

Figure 2.2: Summary diagram

Chapter 3

Spectral theorem for self-adjoint unbounded operators

We fix a separable Hilbert space $\mathscr{H}$. We consider a self-adjoint unbounded operator $(A, D(A))$ on $\mathscr{H}$.

3.1 introduction

We start with the notion of multiplication operator.

Definition 3.1.1. *Let $d \in \mathbb{N}$, B a Borel set on $\mathbb{R}^d$ and μ a measure on B. For a function $a \in L^2_{loc}(B, d\mu; \mathbb{C})$, we define the multiplication operator by a as :*

$$(M_a v)(x) := a(x)v(x) \ , \tag{3.1}$$

for any $v \in D(M_a) := \{v \in L^2(B, d\mu) : av \in L^2(B, d\mu)\}$.

The multiplication operator is crucial for spectral theorem. In fact, the idea behind the spectral theorem is to show in a precise way that the operator A can be diagonalized. In our settings, we mean it can be represented as a multiplication operator by a certain function a. As in the case matrices, the function a of multiplication encodes the spectrum of the operator A.

Exercise 3.1.2. *With the notation of Definition 3.1, Show that*

$$\sigma(M_a) = Imess\ a := \left\{\lambda \in \mathbb{C} : \forall \varepsilon > 0,\ \mu(\{x : |\lambda - a(x)| \leq \varepsilon\}) > 0\right\} \ . \tag{3.2}$$

The set Imess a is called the essential image of the function a with respect to the measure μ.

In this chapter, our goal is to prove the following result.

Theorem 3.1.3 (Spectral theorem)**.** *There exist :*

- *a Borel set $B \in \mathbb{R}^d$, with $d \in \mathbb{N}$,*
- *a finite Borel measure μ on B,*
- *a real-valued locally bounded function $a \in L^\infty_{loc}(B, d\mu)$,*
- *a unitary isomorphism $U : H \to L^2(B, d\mu)$*

such that

$$A = U^{-1} M_a U \qquad , \qquad UD(A) = D(M_a) \ . \tag{3.3}$$

Remark 3.1.4. *In fact, we prove that we can take $d = 2$, $B = \sigma(A) \times \mathbb{N} \subset \mathbb{R}^2$ and $a(x, n) = x$ for any $(x, n) \in B$.*

We proceed in three steps:

1. We perform the diagonalization on particular class of subspaces of $\mathscr{H}$ called cyclic spaces.
2. Then, we decompose $\mathscr{H}$ as an orthogonal sum of such cyclic spaces.
3. At the end, we perform a block diagonalization .

Domain issues is omnipresent when dealing with unbounded operators. Thus, to prove Theorem 3.1.3 , we work with the resolvents which are bounded operators. In fact, diagonalizing the resolvents is equivalent to diagonalize the operator itself. We use the structural relationship of the resolvents with functional calculus and spectral measures. It is the cornerstone of the proof. Before the proof, we give a property on the support of spectral measures.

Proposition 3.1.5. *Let $u \in \mathscr{H}$. We consider μ_u^A the spectral measure (2.6) associated to u and A. Then,*

$$supp\ \mu_u^A \subset \sigma(A) \ . \tag{3.4}$$

Proof. Let $\lambda \in \text{supp}\ \mu_u^A$. We know by definition of the support that for all $\varepsilon > 0$, $\mu_u^A([\lambda - \varepsilon, \lambda + \varepsilon]) > 0$. Then, due to (2.9), we have

$$\langle \mathbf{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(A)u, u\rangle = \mu_u^A([\lambda - \varepsilon, \lambda + \varepsilon]) > 0 \ .$$

Thus, we have $\mathbf{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(A)u \neq 0$. We consider

$$v_\varepsilon := \frac{1}{\|\mathbf{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(A)u\|} \mathbf{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(A)u \ .$$

We can observe that $\|v_\varepsilon\| = 1$ and $\mathbf{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(A)v_\varepsilon = v_\varepsilon$. Moreover, v_ε is a Weyl sequence. Indeed,

$$\begin{aligned}\|(A-\lambda)v_\varepsilon\|^2 &= \|(A-\lambda)\mathbf{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(A)v_\varepsilon\|^2 \\ &= \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (x-\lambda)^2 d\mu^A_{v_\varepsilon}(x) \\ &\leq \varepsilon^2 \int_{\mathbb{R}} d\mu^A_{v_\varepsilon}(x) \ .\end{aligned}$$

Due to (2.7), we have

$$\int_{\mathbb{R}} d\mu^A_{v_\varepsilon}(x) = \mu_{v^A_\varepsilon}(\mathbb{R}) = \|v_\varepsilon\|^2 = 1 \ .$$

We obtain

$$\|(A-\lambda)v_\varepsilon\|^2 \leq \varepsilon^2 \ .$$

$\square$

3.2 Proof of Theorem 3.1.3

Step I: Diagonalization on cyclic spaces. Let $v \in \mathscr{H}$. We consider

$$\chi_v := \overline{\{f(A)v : f \in \mathscr{C}_0(\mathbb{R},\mathbb{C})\}}^{\|\cdot\|} \ . \tag{3.5}$$

We call χ_v the cyclic space generated by v. One can note that $v \in \chi_v$. Due to (2.6), we have for any $f \in \mathscr{C}_0(\mathbb{R},\mathbb{C})$,

$$\|f(A)v\|^2 = \int_{\mathbb{R}} |f(x)|^2 d\mu^A_v(x) \ .$$

Since μ^A_v is supported on $\sigma(A)$ (Proposition 3.1.5), we have

$$\|f(A)v\|^2 = \int_{\sigma(A)} |f(x)|^2 d\mu^A_v(x) \ .$$

Thus, we have a unitary map

$$\mathscr{C}_0(\mathbb{R},\mathbb{C}) \to \mathscr{H} \tag{3.6}$$
$$f \mapsto f(A)v \ . \tag{3.7}$$

This map extends to a unitary isomorphism

$$U_v^{-1} : L^2(\sigma(A), d\mu^A_v) \to \chi_v \ . \tag{3.8}$$

We observe that $(A-i)^{-1}\chi_v \subset \chi_v$. Indeed, for any $f \in \mathscr{C}_0$, we have by functional calculus

$$(A-i)^{-1} f(A) v = [\underbrace{x \mapsto (x-i)^{-1} f(x)}_{\in \mathscr{C}_0}](A) v \ .$$

Passing to the limit gives the stability result claimed. Moreover, the following diagram

$$\begin{tikzcd} \chi_v \arrow[r, "(A-i)^{-1}"] \arrow[d, "U_v"'] & B \\ L^2(\sigma(A), d\mu_v^A) \arrow[r, "x \mapsto (x-i)^{-1}"'] & DL^2(\sigma(A), d\mu_v^A) \arrow[u, "U_v^{-1}"'] \end{tikzcd}$$

is commutative. This yields the diagonalization

$$(A-i)^{-1} = U_v^{-1} M_{(x-i)^{-1}} U_v \ . \tag{3.9}$$

Step II: Cyclic decomposition of $\mathscr{H}$.

Lemma 3.2.1. *There exist a family of vectors $v_n \in \mathscr{H}$ such that*

$$\mathscr{H} = \bigoplus_{n \in \mathbb{N}} \chi_{v_n} \qquad \text{and} \qquad \sum_{n \in \mathbb{N}} \|v_n\|^2 < \infty \ .$$

The direct sum is orthogonal.

Proof. Let $(e_n)_{n \in \mathbb{N}}$ be a Hilbert basis of $\mathscr{H}$. we set $v_0 := e_0$. Thus, we have the orthogonal decomposition

$$\mathscr{H} = \chi_{v_0} \oplus \chi_{v_0}^{\perp} \ .$$

We choose $v_1 := \Pi_{\chi_{v_0}^{\perp}} e_1$. Then, we have $\chi_{v_1} \in \chi_{v_0}^{\perp}$. Indeed, for any $f, g \in \mathscr{C}_0$,

$$\langle f(A) v_1, g(A) v_0 \rangle = \langle v_1, \underbrace{(\overline{f} g)(A) v_0}_{\in \chi_{v_0}} \rangle = 0,$$

because $v_1 \in \chi_{v_0}^{\perp}$. Thus, we can write

$$\mathscr{H} = \chi_{v_0} \oplus \chi_{v_1} \oplus (\chi_{v_0} \oplus \chi_{v_1})^{\perp}$$

and in particular $e_1 \in \chi_{v_0} \oplus \chi_{v_1}$. We construct for any $n \geq 1$

$$v_n := \Pi_{(\chi_{v_0} \oplus \cdots \oplus \chi_{v_{n-1}})^{\perp}} e_n \ .$$

By induction, we have

$$\mathscr{H} = \chi_{v_0} \oplus \cdots \oplus \chi_{v_n} \oplus (\chi_{v_0} \oplus \cdots \oplus \chi_{v_n})^{\perp} \ .$$

In particular, $e_n \in \chi_{v_0} \oplus \cdots \oplus \chi_{v_n}$. Then,

$$\text{span}\,\{e_n : n \in \mathbb{N}\} \subset \bigoplus_{n\in\mathbb{N}} \chi_{v_n}\ .$$

This yields the decomposition. Moreover, since $\chi_{v_n} = \chi_{\lambda v_n}$ for any $\lambda \in \mathbb{C}$, we can choose v_n such that the series $\sum_{n\in\mathbb{N}} \|v_n\|^2$ converges. □

Step III: Diagonalization on $\mathscr{H}$. Let $v \in \mathscr{H}$. Due to Lemma 3.2.1, we can write

$$v = \sum_{n\in\mathbb{N}} \langle v, \Pi_{\chi_{v_n}} v\rangle \Pi_{\chi_{v_n}} v\ .$$

Since $(A-i)^{-1}$ is bounded, we have

$$(A-i)^{-1}v = \sum_{n\in\mathbb{N}} \langle v, \Pi_{\chi_{v_n}} v\rangle (A-i)^{-1}\Pi_{\chi_{v_n}} v\ . \tag{3.10}$$

From *Step I*, we know that we can diagonalize $(A-i)^{-1}$ on χ_{v_n}. Thus, we can write

$$(A-i)^{-1}v = \sum_{n\in\mathbb{N}} \langle v, \Pi_{\chi_{v_n}} v\rangle U_{v_n}^{-1} M_{(x-i)^{-1}} U_{v_n} \Pi_{\chi_{v_n}} v\ . \tag{3.11}$$

We set $B := \sigma(A) \times \mathbb{N}$. We consider on B the Borel measure μ (Exercise 3.2.2) defined by

$$\mu(\Omega \times \{n\}) = \mu^A_{v_n}(\Omega)\ , \qquad \text{for any Borel set } \Omega \subset \sigma(A) \text{ and } n \in \mathbb{N}\ . \tag{3.12}$$

This measure is finite due to Lemma 3.2.1. Indeed, we have

$$\mu(\sigma(A)\times\mathbb{N}) = \sum_{n\in\mathbb{N}} \mu^A_{v_n}(\sigma(A)) = \sum_{n\in\mathbb{N}} \|v_n\|^2 < \infty\ .$$

Then, we construct the map

$$\begin{aligned} U : \mathscr{H} = \bigoplus_{n\in\mathbb{N}} \chi_{v_n} &\to L^2(B, d\mu) \\ v &\mapsto \Big(x \mapsto \langle v, \Pi_{\chi_{v_n}} v\rangle \big(U_{v_n}\Pi_{\chi_{v_n}} v\big)(x)\Big)_{n\in\mathbb{N}} \end{aligned}$$

We claim that U is a unitary isomorphism. Indeed, the operator U is a block-diagonal operator whose diagonals are the U_{v_n}.The claim follows from the fact that each operator $U_{v_n} : \chi_{v_n} \to L^2(\sigma(A), d\mu^A_{v_n})$ is a unitary isomorphism. In particular, its inverse is given by

$$\begin{aligned} U^{-1} : L^2(B, d\mu) &\to \mathscr{H} \\ (f_n)_{n\in\mathbb{N}} &\mapsto \sum_{n\in\mathbb{N}} U^{-1}_{v_n} f_n \ . \end{aligned}$$

We define the function $a(x, n) = x$ for any $(x, n) \in B$. Thus, we succeed in diagonalizing the resolvent. In fact, from relation (3.11), we obtain

$$(A - i)^{-1} v = U^{-1} M_{(a-i)^{-1}} U v \ . \tag{3.13}$$

Now, we diagonalize A. We know that Ran $(A - i)^{-1} = D(A)$. On one hand, due to (3.13), we have

$$U D(A) = \{M_{(a-i)^{-1}} U v : v \in \mathscr{H}\} \ .$$

On the other hand,

$$\begin{aligned} f \in D(M_a) &\iff af \in L^2(B, d\mu) \ , \\ &\iff (a - i)f \in L^2(B, d\mu) \quad \text{since } f \in L^2(B, d\mu) \ , \\ &\iff \exists v \in \mathscr{H} \text{ such that } (a - i)f = Uv \ , \\ &\iff \exists v \in \mathscr{H} \text{ such that } f = M_{(a-i)^{-1}} U v \ . \end{aligned}$$

Therefore $UD(A) = D(M_a)$. Moreover, we define the operator

$$\begin{cases} T_a := U^{-1} M_{a-i} U \\ D(T_a) := U^{-1} D(M_a) = D(A) \end{cases}$$

By (3.13), we have

$$(A - i)^{-1} T_a = Id_{D(A)} \quad \text{and} \quad T_a (A - i)^{-1} = Id_{\mathscr{H}} \ . \tag{3.14}$$

By uniqueness of the inverse, we deduce that $T_a = A - i$ and the diagonalization follows.

$\square$

Exercise 3.2.2. *Justify the fact the measure defined at (3.12) is a Borel measure on B.*

3.3 Functional calculus via spectral theorem

Once spectral theorem is established, we can construct a $\mathscr{L}^\infty_{\mathrm{loc}}$ functional calculus. Using the notation of Theorem 3.1.3, for any $f \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$, we define

$$f(A) := U^{-1} M_{f(a)} U \quad \text{ with } D(f(A)) := U^{-1} D(M_{f(a)}) \ . \tag{3.15}$$

Exercise 3.3.1. *Justify that $f(A)$ satisfies (ii) of Proposition 2.4.1.*

But, there is a problem a priori. In fact, this functional calculus could depend on the way we diagonalize A through the choice of U, a, B, μ. The following lemma resolves this issue and shows that the functional calculus (3.15) is the same as the one defined at (2.13).

Lemma 3.3.2. *Let $f \in \mathscr{L}^\infty_{loc}(\mathbb{R}, \mathbb{C})$ and $u \in \mathscr{H}$. We recall the space D_f (see (2.12)). We consider $f_n := f\mathbf{1}_{f \leq n}$. The following holds.*

(i)

$$u \in D_f \iff \limsup_{n \to +\infty} \|f_n(A)u\| < \infty \ .$$

(ii)

$$u \in D(f(A)) \iff \limsup_{n \to +\infty} \|f_n(A)u\| < \infty$$

Moreover, if $u \in D(f(A))$, we have $f_n(A)u \to f(A)u$ when $n \to +\infty$.

Proof. We let the proof as an exercise for the reader (*Hint : Monotone convergence*).

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